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MODULES WITH ABSOLUTE ENDOMORPHISM RINGS*

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Dedicated to our friend Luigi Salce on the occasion of his 60th birthday

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ABSTRACT

Eklof and Shelah [8] call an abelian group **absolutely indecomposable** if it is directly indecomposable in every generic extension of the universe. More generally, we say that an R-module is **absolutely rigid** if its endomorphism ring is just the ring of scalar multiplications by elements of R in every generic extension of the universe. In [8] it is proved that there do not exist absolutely rigid abelian groups of size $\geq \kappa(\omega)$, where $\kappa(\omega)$ is the first ω-Erdős cardinal (for its definition see the introduction). A similar result holds for rigid systems of abelian groups. On the other hand, recently Göbel and Shelah [15] proved that for modules of size $< \kappa(\omega)$ this phenomenon disappears. Their result on R_ω-modules (i.e. on R-modules with countably many distinguished submodules) that establishes the existence of 'well-behaving' fully rigid systems of abelian groups of large sizes $< \kappa(\omega)$ will be extended here to a large class of R-modules by proving the existence of modules of any sizes $< \kappa(\omega)$ with endomorphism rings which are absolute. In order to cover rings as general as possible, we utilize a method developed by Brenner, Butler and Corner (see [2, 3, 5]) to reduce the number of distinguished submodules required in the construction from $\aleph_0$ to five.

We give several applications of our results. They include modules over domains with four pairwise comaximal prime elements, and modules over quasi-local rings whose completions contain at least five algebraically independent elements.

1. Introduction

A module M (over any ring R) is said to be **absolutely indecomposable** if it is indecomposable in every generic extension of the universe. Filling a gap in [8] and in Eklof–Mekler [7, pp. 487–492], Göbel and Shelah [15] have established the existence of absolutely indecomposable abelian groups of any size $< \kappa(\omega)$. They also proved similar results for modules over domains R by constructing modules whose endomorphism rings are isomorphic to R and are absolute in the sense that they do not change under generic extensions of the universe.

We wish to extend their results to modules over various classes of commutative rings R as well as to modules with prescribed endomorphism algebras over such rings by establishing the existence of R-modules with various absolute endomorphism rings. Our starting point is the main theorem on R_ω-modules in [15], which was derived by encoding absolutely rigid valuated trees from Shelah [18] into modules; see Theorem 2.1 below.

Let κ be a (finite or infinite) cardinal and R a commutative ring with $1 \neq 0$. (A cardinal κ is viewed as the set of all ordinals $\alpha < \kappa$; in particular, $5 = \{0, 1, 2, 3, 4\}$.) We first recall the definition of R_κ-modules. An **R_κ-module $\mathbf{X}$** is an R-module X along with a family of R-submodules X_i $(i < \kappa)$; in notation: $\mathbf{X} = (X, X_i \mid i < \kappa)$. Furthermore, $\mathbf{X}$ is **a free R_κ-module** if $X, X_i, X/X_i$ $(i < \kappa)$ are all free R-modules. If $\mathbf{X}$ and $\mathbf{Y} = (Y, Y_i \mid i < \kappa)$ are R_κ-modules, then φ is an R_κ-homomorphism, $\varphi \in \mathrm{Hom}_R(\mathbf{X}, \mathbf{Y})$, if $\varphi \in \mathrm{Hom}_R(X, Y)$ and $X_i\varphi \subseteq Y_i$ for all $i < \kappa$. We also write $\mathrm{Hom}_R(\mathbf{X}, \mathbf{X}) = \mathrm{End}_R\, \mathbf{X}$.

A **fully rigid system** of R_κ-modules on the cardinal λ is a family $\mathbf{F}_U$ $(U \subseteq \lambda)$ of R_κ-modules (with U running over the subsets of λ) of cardinality $\leq \lambda$ such that the following holds for subsets U, V of λ:

$$\mathrm{Hom}_R(\mathbf{F}_U, \mathbf{F}_V) = \begin{cases} R \text{ if} & U \subseteq V \\ 0 \text{ if} & U \not\subseteq V. \end{cases}$$

If R on the right is replaced by an R-algebra A, then we say that this family is a **fully A-rigid system** of R_κ-modules.

Surprisingly the first ω-*Erdős cardinal* $\kappa(\omega)$ is the crucial borderline for our algebraic results on R_ω-modules. This large cardinal may not exist in any model of set theory, but if it exists, then it does also in L (see [18]). It is defined as the smallest cardinal κ such that $\kappa \to (\omega)^{<\omega}$ holds, i.e., for every function f from the finite subsets of κ to 2 there exist an infinite subset $X \subset \kappa$ and a function $g : \omega \to 2$ such that $f(Y) = g(|Y|)$ holds for all finite subsets Y of X. The cardinal $\kappa(\omega)$ is known to be strongly inaccessible; see Jech [16, p. 303].

Now we quote the Main Theorem 4.1 from [15].

THEOREM 1.1: *Let R be any commutative ring with $1 \neq 0$ and λ a cardinal such that $\lambda, |R| < \kappa(\omega)$. There exists a fully rigid system $\mathbf{F}_U$ $(U \subseteq \lambda)$ of free R_ω-modules with the following properties:*

(i) *F is of rank λ and $\mathbf{F}_U = (F, F_U, F_i \mid 0 \neq i \in \omega)$ (thus only $F_0 := F_U$ depends on U).*
(ii) *The family $\mathbf{F}_U$ $(U \subseteq \lambda)$ is absolutely fully rigid, i.e., even if the given universe is replaced by a generic extension, the family stays fully rigid.*

Remark 1.2: We will write $F = \bigoplus_{j<\lambda} Rh_j$ for the free R-module above and similarly $F_U = \bigoplus_{j \in U} Rh_j$ as in [15].

We begin our discussion with R_ω-modules and will continue with R_5-modules by making use of a method developed by Brenner, Butler and Corner (see [2, 3, 5]) and utilized by Franzen–Göbel [9, 12] to reduce the number of modules required in the construction from countably many to five (see Theorem 3.2). Our results extend easily to cover modules over faithful R-algebras A; cf. Theorem 4.2.

We conclude the paper with a few applications; see Section 5 below. In these applications the role of distinguished submodules is played by fully invariant submodules that have to be chosen appropriately. The Main Theorem 4.1 in [15] was stated such that it is readily applicable to our situation.

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2. The link to valuated trees and results above and below the border line $\kappa(\omega)$

For an infinite cardinal λ, $T_\lambda = {}^{\omega>}\lambda$ will denote the tree whose n-th level consists of all finite sequences $f : n \to \lambda$ $(n \in \omega)$, and which is partially ordered in the obvious way. By a **subtree** T we mean a subset closed under initial segments. A **homomorphism** between two trees is a map that preserves both levels and initial segments. A tree T together with a map $v : T \longrightarrow \omega$ is an ω-valuated tree (also called ω-colored tree) and a valuated homomorphism $\varphi : T_1 \longrightarrow T_2$ between two valuated trees with valuations v_i $(i = 1, 2)$ is a tree homomorphism which preserves valuations, i.e. $v_2(\eta\varphi) = v_1(\eta)$ for all $\eta \in T_1$. By $\mathrm{Hom}(T_1, T_2)$ we denote the collection of valuated homomorphisms $T_1 \longrightarrow T_2$.

Our starting point, Theorem 1.1, is based on an interesting result about absolutely rigid, valuated trees by Shelah [18].

Theorem 2.1: *Let $\lambda < \kappa(\omega)$ be a cardinal. There is a family T_α $(\alpha \in 2^\lambda)$ of valuated subtrees (of size λ) in $T_\lambda = {}^{\omega>}\lambda$ such that in any generic extension of the universe the following holds for $\alpha, \beta \in 2^\lambda$*

$$\mathrm{Hom}(T_\alpha, T_\beta) \neq \emptyset \Longrightarrow \alpha = \beta.$$

The following remarks are inserted here to clarify absoluteness. A first order formula $\varphi(x, y)$ is called **absolute** if the following holds. If M_2 is a model of set theory, that is an end-extension (i.e. no new elements are added to a set; see [16, p. 654]) of a countable model M_1 of ZFC and $x, y \in M_1$, then $M_1 \vDash \varphi(x, y)$ (is satisfied) if and only if $M_2 \vDash \varphi(x, y)$. Levi established a criterion for (bounded) formulas to be absolute; this is useful in showing that several statements are absolute; cf. Burgess [4, p. 409] and [4, p. 408, Lemma 1.1 and Section 1.5] for its consequences. It follows, e.g., that being an ordinal or a cub is absolute, but neither being a powerset, nor a stationary set is absolute. See also the connection with the language $L_{\infty\omega}$ (the closure of atomic formulas under negation, arbitrary conjunctions and disjunctions over finitely many variables and under existential and universal quantifications of individual variables) in [8, p. 259–260]. Along these lines (using arguments by Silver) it follows that Shelah's chain of trees constitutes an absolutely rigid family of trees. Moreover, the same holds for the modules (as in [15]) with distinguished submodules derived from these trees. However, rigid families of $\aleph_1$-free families of abelian groups (for example) of cardinality $\aleph_1$ are not absolute, because by cardinal collapsing they can be made countable, and hence free in a suitable model of ZFC. Thus small rigid families can only be used to get large absolute families (as in Section 5) if they remain (absolutely) rigid. In contrast to absolute properties of modules we now quote from Eklof–Shelah [8] (see also [18]) the following striking results (whose proofs are surprisingly not so complicated as expected).

The stated property of rigid families of trees in Theorem 2.1 fails whenever $\lambda \geq \kappa(\omega)$, i.e. for any family T_α $(\alpha < \lambda)$ of valuated subtrees there are a generic extension of the universe and distinct ordinals $\alpha, \beta < \lambda$ such that $\mathrm{Hom}(T_\alpha, T_\beta) \neq \emptyset$. In fact, the arguments in [8] apply to any family of structures, so their results can be applied to R-modules. Consequently, we can just quote without proofs

Theorem 2.2 (Eklof–Shelah [8]): *Let λ be a cardinal $\geq \kappa(\omega)$ and $\{M_\nu : \nu < \lambda\}$ a family of non-zero left R-modules, R any ring with 1. Then there are distinct ordinals $\alpha, \beta < \lambda$, such that in some generic extension $V[G]$ of the universe V, there is an injective homomorphism $\phi : M_\alpha \to M_\beta$.*

Theorem 2.3 (Eklof–Shelah [8]): *Let κ be a cardinal $\geq \kappa(\omega)$ and M an R-module of cardinality κ, where R is any commutative ring with 1 such that $\kappa > |R|$. Then there exists a generic extension $V[G]$ of the universe V, such that M has an endomorphism (actually also a monomorphism) that is not multiplication by an element of R.*

Thus, if we wish to construct R-modules with absolute endomorphism rings R, then we should restrict the size of modules under consideration to cardinals below $\kappa(\omega)$.

3. Reduction to five distinguished submodules

It is a remarkable fact that the results in Theorem 1.1 hold even if the category of R_ω-modules is replaced by the category of R_5-modules. This is an important advantage in applications. In order to justify this reduction, we apply results and methods developed in [2, 3, 5].

Let R be a commutative ring with $1 \neq 0$, and let

$$E = Re_1 \oplus Re_2 \oplus Re_3 \quad \text{and} \quad G = \bigoplus_{n<\omega} Rg_n$$

be free R-modules with generators e_1, e_2, e_3 and g_n $(n < \omega)$, respectively. In the tensor product $E \otimes_R G$ choose four distinguished submodules

$$W^1 = Re_1 \otimes G, \quad W^2 = Re_2 \otimes G, \quad W^3 = Re_3 \otimes G$$
$$\text{and} \quad W^4 = R(e_1 + e_2 + e_3) \otimes G.$$

Furthermore, let $\rho, \sigma : \omega \longrightarrow \omega$ be two order preserving maps such that

(i) $\operatorname{Im} \sigma \subset \operatorname{Im} \rho$,

(ii) $\operatorname{Im} \rho \setminus \operatorname{Im} \sigma$ is infinite,

(iii) jump condition: $[\sigma(k), \sigma(k) + k] \cap \operatorname{Im} \rho = \{\sigma(k)\}$ for all $k < \omega$,

where $[\sigma(k), \sigma(k) + k]$ denotes the interval $\{n < \omega : \sigma(k) \leq n \leq \sigma(k) + k\}$. It is easy to construct the functions ρ and σ by induction; the pair ρ, σ remains fixed in what follows. By making use of these functions, we define two additional submodules:

$$W^0 = \bigoplus_{i<\omega} R(e_1 \otimes g_i + e_2 \otimes g_{i+1}) \oplus \bigoplus_{i<\omega} R(e_3 \otimes g_{\sigma(i)}) \subseteq E \otimes G$$

and

$$W'^0 = \bigoplus_{i<\omega} R(e_1 \otimes g_i + e_2 \otimes g_{i+1}) \oplus \bigoplus_{i<\omega} R(e_3 \otimes g_{\rho(i)}) \subseteq E \otimes G.$$

From condition (i) we obtain $W^0 \subseteq W'^0$, while (ii) implies that W'^0/W^0 is a free R-module of countable rank.

We start with the following lemma, using the six submodules of $E \otimes G$ just introduced; its proof is a modification of the proof of Proposition 3.1 in Göbel [12]. (We are going to use only conditions (i) and (iii).)

LEMMA 3.1: *Let $R \neq 0$ be a commutative ring, and let X, Y be R-modules. Then the R_5-modules*

$$\mathbf{C}_X = (C_X = E \otimes G \otimes X,\ C_X^j = W^j \otimes X \mid j = 0, 1, 2, 3, 4),$$

$$\mathbf{C'}_Y = (C'_Y = E \otimes G \otimes Y,\ C'^0_Y = W'^0 \otimes Y,\ C'^j_Y = W^j \otimes Y \mid j = 1, 2, 3, 4)$$

satisfy

$$\mathbf{Hom}_R(\mathbf{C}_X, \mathbf{C'}_Y) = \mathbf{1}_E \otimes \mathbf{1}_G \otimes \mathrm{Hom}_R(X, Y).$$

Proof. In order to verify the last equality, it is enough to show that the left hand side is contained in the right hand side. So assume $\phi \in \mathbf{Hom}_R(\mathbf{C}_X, \mathbf{C'}_Y)$. From the invariance of the subspaces W^j $(j = 1, 2, 3, 4)$ we conclude that there exist $\phi_j \in \mathrm{Hom}_R(G \otimes X, G \otimes Y)$ $(j = 1, 2, 3, 4)$ such that

$$(e_j \otimes g \otimes x)\phi = e_j \otimes (g \otimes x)\phi_j, \quad j = 1, 2, 3$$

and

$$[(e_1 + e_2 + e_3) \otimes g \otimes x]\phi = (e_1 + e_2 + e_3) \otimes (g \otimes x)\phi_4$$

for all $g \in G, x \in X$. Comparison yields $\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi'$, thus

$$\phi = \mathbf{1}_E \otimes \phi' \quad \text{with } \phi' \in \mathrm{Hom}_R(G \otimes X, G \otimes Y).$$

We note that

$$W'^0 \cap (W^1 \oplus W^2) = W^0 \cap (W^1 \oplus W^2) = \bigoplus_{j<\omega} R(e_1 \otimes g_j + e_2 \otimes g_{j+1}),$$

and therefore, for $x \in X$ we have $(e_1 \otimes g_i + e_2 \otimes g_{i+1}) \otimes x \in (W^0 \cap (W^1 \oplus W^2)) \otimes X$ and

$$e_1 \otimes (g_i \otimes x)\phi' + e_2 \otimes (g_{i+1} \otimes x)\phi' \in \bigoplus_{j<\omega} R(e_1 \otimes g_j + e_2 \otimes g_{j+1}) \otimes Y.$$

We also note that $G \otimes X = \bigoplus_{i<\omega} Rg_i \otimes X$ and $G \otimes Y = \bigoplus_{i<\omega} Rg_i \otimes Y$. For all $i, j < \omega$, there exists a map $\phi_{ij} \in \mathrm{Hom}_R(X, Y)$ such that

$$(g_i \otimes x)\phi' = \sum_{j<\omega} (g_j \otimes x\phi_{ij}).$$

Hence there are elements $y_{ij} \in Y$ $(i, j < \omega)$ such that

$$\begin{aligned} e_1 \otimes \sum_{j<\omega}(g_j \otimes x\phi_{ij}) + e_2 \otimes \sum_{j<\omega}(g_j \otimes x\phi_{i+1,j}) &\\ = e_1 \otimes (g_i \otimes x)\phi' + e_2 \otimes (g_{i+1} \otimes x)\phi' &\\ = \sum_{j<\omega}(e_1 \otimes g_j + e_2 \otimes g_{j+1}) \otimes y_{ij} &\\ = \sum_{j<\omega}(e_1 \otimes g_j) \otimes y_{ij} + \sum_{j<\omega}(e_2 \otimes g_{j+1}) \otimes y_{ij}. \end{aligned}$$

We get

$$\sum_{j<\omega}(e_1 \otimes g_j) \otimes (x\phi_{ij} - y_{ij}) + \sum_{j\geq 1}(e_2 \otimes g_j) \otimes (x\phi_{i+1,j} - y_{i,j-1}) + (e_2 \otimes g_0) \otimes x\phi_{i+1,0} = 0,$$

whence equating coefficients we obtain $x\phi_{ij} = y_{ij}, x\phi_{i+1,j} = y_{i,j-1}$ $(j > 0)$ and also $x\phi_{i0} = 0$ $(i \geq 1)$. It follows that $x\phi_{ij} = x\phi_{i+1,j+1}$, whence $g_j \otimes x\phi_{ij} = g_j \otimes x\phi_{0,j-i}$ whenever $j \geq i$; also $x\phi_{ij} = 0$ if $j < i$. There is a $k < \omega$ such that

$$(g_i \otimes x)\phi' = \sum_{j<\omega} g_j \otimes x\phi_{ij} = \sum_{j\leq k} g_{i+j} \otimes x\phi_{0j}. \tag{3.1}$$

We now keep the element $x \in X$ fixed and note that in view of the definition of ϕ' the set $\{j < \omega : x\phi_{0j} \neq 0\}$ must be finite. Thus k in the formula (3.1) is the largest element of this set and does not depend on i.

Finally, we observe that

$$W'^0 \cap W^3 = \bigoplus_{i<\omega} R(e_3 \otimes g_{\rho(i)}) \quad \text{and} \quad W^0 \cap W^3 = \bigoplus_{i<\omega} R(e_3 \otimes g_{\sigma(i)}),$$

thus, for each $k < \omega$, $e_3 \otimes g_{\sigma(k)} \otimes x \in W^0 \cap W^3 \otimes X$ and

$$(e_3 \otimes g_{\sigma(k)} \otimes x)\phi \in \bigoplus_{i<\omega} R(e_3 \otimes g_{\rho(i)}) \otimes Y.$$

If we pick k as in equation (3.1), then we can find elements $y_j \in Y$ such that

$$\begin{aligned} e_3 \otimes \sum_{j<\omega}(g_{\rho(j)} \otimes y_j) = (e_3 \otimes g_{\sigma(k)} \otimes x)\phi = e_3 \otimes (g_{\sigma(j)} \otimes x)\phi' &\\ = e_3 \otimes \sum_{j\leq k}(g_{\sigma(k)+j} \otimes x\phi_{0j}). \end{aligned}$$

We have $\sum_{j<\omega} g_{\rho(j)} \otimes y_j = \sum_{j\leq k} g_{\sigma(k)+j} \otimes x\phi_{0j}$, and by the jump condition (iii) it follows that $x\phi_{0i} = 0$ for $1 \leq i \leq k$. Hence also $x\phi_{0i} = 0$ for all $x \in X$.

We get $\phi_{0i} = 0$ for all $0 \neq i < \omega$, and if we put $\phi_{00} = \psi$, then we obtain $(g_i \otimes x)\phi' = g_i \otimes x\psi$ and $\phi = \mathbf{1}_E \otimes \mathbf{1}_G \otimes \psi$, as desired. ∎

For a cardinal $\lambda < \kappa(\omega)$, consider the (fixed) fully rigid system

$$\mathbf{F}_U = (F, F_0 = F_U, F_i \mid 0 < i < \omega)$$

of R_ω-modules (with $U \subseteq \lambda$) whose existence is guaranteed by Theorem 1.1, where $R \neq 0$ is any commutative ring with $|R| < \kappa(\omega)$. For each $U \subseteq \lambda$, we define an R_5-module

$$\mathbf{N}_U = (N, N_0 = N_U, N_j \mid \; j = 1, 2, 3, 4)$$

in the following way.

Let E, G be free R-modules as defined above, and let

$$N = E \otimes G \otimes F$$

with the following five distinguished submodules. Choose an infinite set $B = \{r_i : i < \omega\} \subset \operatorname{Im}\rho \setminus \operatorname{Im}\sigma$, and set

$$N_j = W^j \otimes F, \quad j = 1, 2, 3, 4$$

and

$$N_U = \bigoplus_{i<\omega}(R(e_1 \otimes g_i + e_2 \otimes g_{i+1}) \otimes F) \oplus \bigoplus_{i<\omega}(R(e_3 \otimes g_{\sigma(i)}) \otimes F) \oplus \bigoplus_{i<\omega}(R(e_3 \otimes g_{r_i}) \otimes F_i).$$

Thus we have the inclusion relations $W^0 \otimes F \subset N_U \subset W'^0 \otimes F$.

We would like to point out that all crucial information about rigidity of the R_5-modules just constructed is hidden in the first invariant submodule N_0. This is reflected in the proof of our next claim.

Theorem 3.2: *Assume $R \neq 0$ is a commutative ring and λ a cardinal such that $|R| \leq \lambda < \kappa(\omega)$. Let $\mathbf{N}_U = (N, N_j \mid j = 0, 1, 2, 3, 4)$ ($U \subseteq \lambda$) (with $N_0 = N_U$) be a set of R_5-modules just defined. If X, Y are any faithful R-modules, then in any generic extension of the universe*

$$\operatorname{Hom}_R(\mathbf{N}_U \otimes X, \mathbf{N}_V \otimes Y) = \begin{cases} \mathbf{1}_U \otimes \operatorname{Hom}(X, Y) & \text{if} \quad U \subseteq V \\ 0 & \text{if} \quad U \not\subseteq V. \end{cases}$$

Proof. We consider an arbitrary R_5-homomorphism $\chi : \mathbf{N}_U \otimes X \to \mathbf{N}_V \otimes Y$, where $U, V \subseteq \lambda$. Thus $\chi : E \otimes G \otimes F \otimes X \to E \otimes G \otimes F \otimes Y$ satisfies $(N_j \otimes X)\chi \subseteq N_j \otimes Y$ for $j = 1, 2, 3, 4$ and $(N_U \otimes X)\chi \subseteq N_V \otimes Y$. An appeal to

Lemma 3.1 where we let X be $F \otimes X$ and Y be $F \otimes Y$ shows that $\chi = \mathbf{1}_{E \otimes G} \otimes \varphi$, where $\varphi \in \mathrm{Hom}(F \otimes X, F \otimes Y)$. Next we exploit the special choice of N_U and N_V, respectively. It follows that $(e_3 \otimes g_{r_i} \otimes F_i \otimes X)\chi \subseteq (e_3 \otimes g_{r_i} \otimes F_i \otimes Y)$ for all $i < \omega$. Thus $F_i \otimes X$ is mapped by φ into $F_i \otimes Y$ for each $i < \omega$; equivalently, φ is an R_ω-homomorphism

$$(F \otimes X, F_0 \otimes X = F_U \otimes X, F_i \otimes X \mid 0 < i < \omega) \to$$
$$(F \otimes Y, F_0 \otimes Y = F_U \otimes Y, F_i \otimes Y \mid 0 < i < \omega).$$

The conclusion now follows at once from Theorem 1.1. ∎

In the important special case $X = Y = R$, we have

Corollary 3.3: *For every commutative ring $R \neq 0$ and for every cardinal $\lambda < \kappa(\omega)$ with $|R| \leq \lambda$, there exists an R_5-module $\mathbf{M}$ of cardinality λ such that $\mathrm{End}\,\mathbf{M} = R$ in every generic extension of the universe.*

Proof. This follows from the preceding theorem. Indeed, let $\mathbf{M} = \mathbf{N}_U \otimes R$; then by the theorem $\mathrm{End}\,\mathbf{M} = \mathrm{Hom}(R, R)$ and $\mathrm{Hom}(R, R) = R$ in any generic extension of the universe. ∎

In particular, if $R = K$ is a field, then (with the choice $X \cong K$) the preceding result confirms the existence of absolutely indecomposable K_5-vector spaces of sizes $< \kappa(\omega)$. It is likely that the same holds for K_4-vector spaces, but it is certain that the result cannot be improved to K_3-vector spaces. Indeed, K_3-vector spaces are of finite representation type (see Simson [20]), and from a theorem by Ringel–Tachikawa [17] (see also Simson [19]) it follows that all infinite dimensional K_3-vector spaces decompose; cf. also Böttinger-Göbel [1, Theorem 4.1].

4. Extension to R-algebras

Our next purpose is to extend the results from R-modules to A-modules, where A is an R-algebra. We are going to apply a well-known method to construct a particular R-module X^* which encodes scalar multiplications by elements from A, see [2, 3]. We will adjoin this additional distinguished submodule to the countably many distinguished submodules considered so far. It will guarantee that the R-homomorphisms will become automatically A-homomorphisms.

Lemma 4.1: *Let $\kappa \leq \lambda$ be cardinals and A be an R-algebra which is generated by at most κ elements (as an algebra) and let X, Y be (right) A-modules. If H is a free R-module of rank λ, then there are submodules $X^* \subseteq H \otimes X$ and $Y^* \subseteq H \otimes Y$, R-isomorphic to $\bigoplus_\kappa X$ and $\bigoplus_\kappa Y$, respectively, such that*

$$\mathrm{Hom}_A(X, Y) = \big\{\varphi \in \mathrm{Hom}_R(X, Y) : X^*(\mathbf{1}_H \otimes \varphi) \subseteq Y^*\big\}.$$

Proof. Let $\{a_i : i < \kappa\}$ be a set generating A as an R-algebra. Choose two disjoint subsets of the free generators of H of size κ and label these elements as h_i, h'_i $(i < \kappa)$. If $i < \kappa$, then $X_i := \{h_i \otimes x + h'_i \otimes xa_i : x \in X\}$ is a summand of $H \otimes X$ isomorphic to X, hence $X^* = \bigoplus_{i<\kappa} X_i$ is a canonical summand of $H \otimes X$ which is isomorphic to $\bigoplus_\kappa X$. Similarly we define Y^* as a submodule of $H \otimes Y$.

If $\psi \in \mathrm{Hom}_A(X, Y)$, then it is immediate that $X^*(\mathbf{1}_H \otimes \psi) \subseteq Y^*$. It remains to show that any $\varphi \in \mathrm{Hom}_R(X, Y)$ with $X^*(\mathbf{1}_H \otimes \varphi) \subseteq Y^*$ is an A–homomorphism. For $x \in X$ we can find $y \in Y$ such that

$$(h_i \otimes x + h'_i \otimes xa_i)(\mathbf{1}_H \otimes \varphi) = h_i \otimes (x\varphi) + h'_i \otimes (xa_i)\varphi = h_i \otimes y + h'_i \otimes ya_i.$$

Thus $x\varphi = y, (xa_i)\varphi = ya_i = (x\varphi)a_i$ for all $i < \kappa$, and φ is an A-homomorphism. ∎

Using R_5–modules in place of R–modules we are able to adopt arguments from [9] which were used for the construction of indecomposable modules by the Shelah elevator, see also [2, 3, 5] for related arguments. The following theorem will become a consequence of Theorem 1.1 and Lemma 4.1. Observe that we will consider R-homomorphisms between A-modules. The preceding lemma will be used to represent R-algebras A which are generated by κ $(\leq \lambda)$ elements. We combine it with Theorem 1.1 and Lemma 3.1.

First, we construct the R_5-modules we will need in Theorem 4.2. Let $\lambda < \kappa(\omega)$ and $U \subseteq \lambda$ a subset of λ. Then we fix $F = \bigoplus_{i<\lambda} Rh_i$ from Remark 1.2 and choose $F_U = \bigoplus_{i\in U} Rh_i \subseteq F$. If X is a module over the R-algebra A, then we define the R_5-module

$$\mathbf{M}_{UX} = (M_X, M^0_{UX}, M^j_X : 1 \leq j \leq 4)$$

by making use of the modules in Lemma 3.1 as follows. Set

$$M_X = E \otimes G \otimes (F \otimes X) \quad \text{and} \quad M^j_X = W^j \otimes (F \otimes X) \quad \text{for } j = 1, 2, 3, 4.$$

(Recall from Section 3 that $W^j \subseteq E \otimes G$.) It remains to define M^0_X. By the definitions of W^0 and W'^0 we have $W'^0 = W^0 \oplus C \subseteq E \otimes G$ for some free R-module C of countable rank (recall condition (ii) in Section 2). Thus with $a_n \in E, b_1, b_2 \in E \otimes G$ and $G' \subseteq G$ we can write

$$C = \bigoplus_{n<\omega} (Ra_n \otimes G') \oplus Rb_1 \oplus Rb_2,$$

where G' is a copy of G (because the displayed module has only countable rank). If $\sigma : G \longrightarrow G'$ is an R-isomorphism, then the isomorphism $\sigma' = \sigma \otimes \mathbf{1}_F : G \otimes F \longrightarrow G' \otimes F$ turns $F' := G' \otimes F$ into an R_ω-module using $F'_i := (G \otimes F_i)\sigma'$ as the distinguished submodules coming from Theorem 1.1.

Now we define M^0_X such that $M^0_X \subseteq M^0_{UX} \subseteq M'^0_X$. Consider $M^0_X := W^0 \otimes F \otimes X$, and let

$$M'^0_X := M^0_X \oplus (C \otimes F \otimes X).$$

We now set

$$M^0_{UX} = M^0_X \oplus \bigoplus_{i<\omega} (Ra_i \otimes F'_i \otimes X) \oplus Rb_1 \otimes X^* \oplus Rb_2 \otimes F_U \otimes X \tag{4.1}$$

(where $X^* \subseteq F \otimes X$ comes from Lemma 4.1 (for $F = H$) and the summand in brackets in (4.1) is a submodule of $C \otimes F \otimes X$); thus $M^0_X \subseteq M^0_{UX} \subseteq M'^0_X$ follows. If Y is another A-module and $V \subseteq \lambda$, then

$$\varphi \in \mathrm{Hom}(\mathbf{M}_{UX}, \mathbf{M}_{VY})$$

is an R–homomorphism $E \otimes G \otimes F \otimes X \longrightarrow E \otimes G \otimes F \otimes Y$ such that $M^j_X \varphi \subseteq M^j_Y$ for $j = 1, 2, 3, 4$ and $M^0_X \varphi \subseteq M'^0_Y$. (Note that we provided enough space in M'^0_Y to ensure that the last inclusion holds.) By Lemma 3.1 there is $\psi \in \mathrm{Hom}_R(F \otimes X, F \otimes Y)$ such that $\varphi = \mathbf{1}_{E \otimes G} \otimes \psi$. From $G' \subseteq G$ it follows

$$(G' \otimes F \otimes X)\varphi = (G' \otimes F \otimes X)(\mathbf{1}_{E \otimes G} \otimes \psi) = G' \otimes (F \otimes X)\psi \subseteq G' \otimes F \otimes Y.$$

Hence $(F'_i \otimes X)\varphi \subseteq F'_i \otimes Y$ for all $n < \omega$. Now Theorem 1.1 applies to $\varphi' = \varphi \restriction G' \otimes F \otimes X$ and $\varphi' = \mathbf{1}_{G' \otimes F} \otimes \psi'$ holds for some $\psi' \in \mathrm{Hom}_R(X, Y)$. We get $\varphi = \mathbf{1}_{E \otimes G \otimes F} \otimes \psi'$. Finally $b_1 \in G'$ and $Rb_1 \otimes X^* \subseteq Rb_1 \otimes F \otimes X$ imply $X^*\psi \subseteq Y^*$. By Lemma 4.1, $\psi' \in \mathrm{Hom}_A(X, Y)$ and $\varphi = \mathbf{1}_{E \otimes G \otimes F} \otimes \psi'$ is an A-homomorphism. If $U \not\subseteq V$, then $\varphi = 0$ is immediate. We thus derived the following general result for A-modules X, Y.

$$\operatorname{Hom}_R(\mathbf{M}_{UX}, \mathbf{M}_{VY}) = \begin{cases} \mathbf{1}_{E\otimes G\otimes F} \otimes \operatorname{Hom}_A(X, Y) \text{ if } & U \subseteq V \\ 0 \text{ if } & U \not\subseteq V. \end{cases}$$

If we put $X = Y = A$, drop A and write

$$\mathbf{M}_{UA} = \mathbf{M}_U = (M, M_U^0, M^j \mid 1 \le j \le 4),$$

this implies the following

THEOREM 4.2: *Let $\lambda < \kappa(\omega)$ be any infinite cardinal and A any faithful algebra over the commutative ring $R \neq 0$ such that A has at most λ generators over R. Then there exists a family of free right A_5-modules*

$$\mathbf{M}_U = (M, M_U^0, M^1, M^2, M^3, M^4) \quad (U \subseteq \lambda),$$

where $M, M_U^0, M^j, M/M_U^0, M/M^j$ are free A-modules of rank λ for all $1 \le j \le 4$ such that

$$\operatorname{Hom}_R(\mathbf{M}_U, \mathbf{M}_V) = \begin{cases} A \textit{ if } & U \subseteq V \\ 0 \textit{ if } & U \not\subseteq V. \end{cases}$$

holds in any generic extension of the universe. ∎

Remark 4.3: We note that if we form the modules $M_U = M_{UA'}$ in Theorem 4.2 for an algebra extension A' of A rather than for A, then the arguments above lead to a strengthening of Theorem 4.2, as noticed and shown in [14, pp. 38–41] in a parallel case:

Let $A \subseteq A'$ be an R-algebra extension satisfying the cardinality condition of Theorem 4.2 for A' and let $\mathbf{M}_U$ be the A_5-module from above, then

$$\mathbf{M}_U \otimes_R A' = (M \otimes_R A', M_U^0 \otimes_R A', M^1 \otimes_R A', M^2 \otimes_R A', M^3 \otimes_R A', M^4 \otimes_R A'), \quad (U \subseteq \lambda)$$

satisfies Theorem 4.2 with A' in place of A.

5. Passing to R-modules

In this final section we now apply our results to various special cases to claim absolute properties and to strengthen known results in the literature. First we assume that there are enough primes (or other objects needed) in the ring, and finally we discuss other cases, including quasi-local rings.

We start with the case where the R-algebra A admits particular modules.

Definition 5.1: Given an R-algebra A and a natural number n we say that the family

$$(X, X^j, \overline{X} \mid j < n)$$

is **almost fully rigid (for** A**)** if the following properties hold:

(i) $X \subset \overline{X}$ are faithful A-modules;
(ii) $X \subset X^j \subset \overline{X}$ for $j < n$;
(iii) $X = \bigcap_{j<n} X^j$ and $\overline{X} = \sum_{j<n} X^j$;
(iv) $\{\phi \in \mathrm{End}_R \overline{X} : X^j\phi \subseteq X^j \ \forall j < n\} = A$;
(v) $\mathrm{Hom}_R(X^j, X^k) = 0$ if $j, k < n$ and $j \neq k$.

For suitable numbers n and algebras A these modules are used to create distinguished submodules by making them fully invariant. Notice that Definition 5.1 is weaker than a fully rigid system in the sense of Corner [6].

Recall that a family of A-modules M_U ($U \subseteq \lambda$) is A**-rigid** for some R-algebra A if

$$\mathrm{Hom}_R(M_U, M_V) = \begin{cases} A \text{ if} & U \subseteq V \\ 0 \text{ if} & U \not\subseteq V. \end{cases}$$

We will say that it is **absolutely** A**-rigid** if it is A-rigid in any generic extension of the universe. Evidently, the modules in an absolutely A-rigid system are absolutely indecomposable whenever A has no idempotents $\neq 0, 1$.

Several applications are based on the following lemma.

LEMMA 5.2: *Let $\lambda < \kappa(\omega)$ be any infinite cardinal. Suppose $R \neq 0$ is a commutative ring admitting an R-algebra A with a family of A-modules satisfying Definition 5.1 for $n = 5$. For every infinite cardinal λ (for which A is at most λ-generated), there exists an absolutely A-rigid family of A-modules M_U ($U \subseteq \lambda$) of cardinality λ.*

Proof. We use the free right A_5-modules $\mathbf{M}_U = (M, M^0_U, M^1, M^2, M^3, M^4)$ for $U \subseteq \lambda$ as stated above in Theorem 4.2, where $M^j = W^j \otimes (F \otimes X^j) \subseteq E \otimes G \otimes (F \otimes \overline{X})$ with the crucial submodule $M^0_U = M^0_{X^0 U} \subseteq E \otimes G \otimes F \otimes \overline{X}$ from (4.1). Set

$$M_U = M^0_U + \sum_{1 \leq j \leq 4} M^j \subseteq E \otimes G \otimes F \otimes \overline{X}.$$

Next we consider the case $U \subseteq V \subseteq \lambda$, and let ξ be a homomorphism mapping M_U into M_V. From Definition 5.1(v) it follows that ξ has to map each of the submodules M^j into M^j for $1 \le j \le 4$ and similarly M^0_U into M^0_V. We can view ξ as a map

$$(M, M^0_U, M^1, M^2, M^3, M^4) \to (M, M^0_V, M^1, M^2, M^3, M^4).$$

By Theorem 4.2 and Definition 5.1(iv) the map ξ acts as scalar multiplication by an element from A. If $U \not\subseteq V$, then by a similar argument we have $\xi = 0$. Thus $\{M_U : U \subseteq \lambda\}$ is an absolutely A-rigid family. ∎

Next we apply this lemma to special cases. In each case, we have to specify how the algebra A and the A-modules in Definition 5.1 have to be chosen.

Case A: Let R be a domain with at least 4 prime elements p_1, p_2, p_3, p_4 that are pairwise comaximal, i.e. $Rp_j + Rp_k = R$ if $j \ne k$. Consider the following R-submodules of the quotient field Q of R:

$$X^0 = p_1^{-\infty} p_2^{-\infty} R, \quad X^1 = p_1^{-\infty} p_3^{-\infty} R, \quad X^2 = p_1^{-\infty} p_4^{-\infty} R, \quad X^3 = p_2^{-\infty} p_3^{-\infty} R,$$
$$X^4 = p_2^{-\infty} p_4^{-\infty} R$$

(where the symbol $p^{-\infty}$ is an abbreviation for $\bigcup_{k<\omega} p^{-k}$). It is straightforward to see that conditions in Definition 5.1 are satisfied with $A = R = X$, so we can apply the preceding lemma and claim

Corollary 5.3: *If R is a domain with at least 4 pairwise comaximal prime elements, then for every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely R-rigid family of torsion-free R-modules M_U $(U \subseteq \lambda)$ of cardinality λ.*

Case B: Let $R = \mathbb{Z}$ and $X = \mathbb{Z}, \overline{X} = H$, where $H \subset \mathbb{Q}a_0 \oplus \cdots \oplus \mathbb{Q}a_4$ is a rank 5 indecomposable torsion-free group (of Pontryagin type), constructed e.g. as in Fuchs [10, p. 125, Example 5], by using algebraically independent p-adic units π_j $(j \le 4)$. We set $X^j = H \cap (\mathbb{Q}a_0 \oplus \mathbb{Q}a_j)$, and observe that all these groups are indecomposable and homogeneous of type $\mathbb{Z}$; furthermore, their endomorphism rings are $\mathbb{Z}$. It is easily seen that conditions in Definition 5.1 will be satisfied, so we can conclude

Corollary 5.4: *For every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely $\mathbb{Z}$-rigid family of homogeneous torsion-free abelian groups M_U $(U \subseteq \lambda)$ of type $\mathbb{Z}$ and of cardinality λ.*

Let us point out that the preceding two examples can be easily extended to the case where the endomorphism rings of the modules are isomorphic to A, for algebras A considered.

CASE C: Let R be a domain with quotient field Q such that Q/R has a summand of the form

$$\overline{X}/R = X^0/R \oplus X^1/R \oplus X^2/R \oplus X^3/R \oplus X^4/R$$

with non-zero components. From Fuchs-Salce [11, p. 504] it follows that the hypothesis of Definition 5.1 is satisfied with $A = R$ and $X = R$. Hence we can state

COROLLARY 5.5: *If R is an infinite domain such that Q/R has a summand as stated, then for every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely R-rigid family of torsion-free R-modules M_U $(U \subseteq \lambda)$ of cardinality λ.*

The hypotheses of the preceding corollary are satisfied, for instance, by an h-local domain with at least 4 maximal ideals if we modify the construction following the pattern of Case A.

CASE D: Let $R = \mathbb{Z}$ and assume X is a torsion-free abelian group with endomorphism ring A. We may view X as a right A-module. Assume there are four primes p_j $(j < 4)$ for which X is reduced, i.e. $\bigcap_{k<\omega} p_j^k X = 0$ for each $j < 4$. Define

$$X^{k,\ell} = p_k^{-\infty} p_\ell^{-\infty} X$$

as a subgroup in the divisible hull $\mathbb{Q} \otimes X$ of X, where the pair (k, ℓ) ranges over the 2-element subsets of $\{0, 1, 2, 3\}$. Evidently, conditions in Definition 5.1 are satisfied, so we can claim

COROLLARY 5.6: *Let X be a torsion-free abelian group with endomorphism ring A such that X is reduced for at least 4 primes. Then for every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely A-rigid family of torsion-free groups M_U $(U \subseteq \lambda)$ of cardinality λ.*

Let R be a commutative ring and $p \in R$ a non-zerodivisor of R such that the p-adic topology on R is Hausdorff. Assume that A is an R-algebra which is a p-pure submodule between R and its p-adic completion $\widehat{R}$, so that $\widehat{A} = \widehat{R}$. We also assume that $\widehat{R}$ has transcendence degree at least 5 over A. Thus, there exist

$\pi_\gamma \in \widehat{R}$ ($\gamma \in I$) with $|I| \geq 5$ which are algebraically (or at least quadratically) independent over A.

Remark 4.3 ensures the existence of an R_5-module

$$\mathbf{M} = (M, M^0, M^1, M^2, M^3, M^4)$$

of rank $\lambda < \kappa(\omega)$ with $\operatorname{End} \mathbf{M} = A$ such that $\widetilde{\mathbf{M}} = \mathbf{M} \otimes_R \widehat{A}$ satisfies also $\operatorname{End} \widetilde{\mathbf{M}} = \widehat{A}$. Now $\widetilde{M} = M \otimes_R \widehat{A}$, $\widetilde{M^i} = M^i \otimes_R \widehat{A}$, and note that $\widetilde{M} \subseteq \widehat{M}$ holds for the p-adic completion $\widehat{M}$ of M.

Using five algebraically (or quadratically) independent elements $\pi_j \in \widehat{R}$ ($j \leq 4$), we form the R-module

$$H = \langle M, \pi_j M^j \mid j \leq 4 \rangle_* \subseteq \bigoplus_\lambda \widehat{A},$$

the p-purification of $\langle M, \pi_j M^j \mid j \leq 4 \rangle$ in $\bigoplus_\lambda \widehat{A}$.

Theorem 5.7: *For a p-pure subalgebra A of $\widehat{R}$ with the indicated notation,* $\operatorname{End}_R H = A$ *holds absolutely.*

Proof. Let $H_j = \pi_j M^j_*$ be the purification of M^j in $\widetilde{M}$ for $j \leq 4$. If $h \in H \cap \pi_j H$, then there are relations $p^n h = x + \sum_i \pi_i x_i = \pi_j(x' + \sum_i \pi_i x_i')$ for some $x, x_i, x', x_i' \in M$ and $n \in \mathbb{N}$. Thus $x + \sum_i \pi_i x_i = \pi_j x' + \sum_i \pi_j \pi_i x_i'$. Recall that M is a free A-module, thus (looking at restrictions of the last equation to A-summands) the quadratic independence of the π_is over A guarantees that $x_i' = 0$ for all i. Hence the last equation becomes $x + \sum_i \pi_i x_i = \pi_j x'$, which implies that $x = 0, x_i = 0$ for all $i \neq j$ and $x_j = x'$. We derive $p^n h = \pi_j x' \in M^j$ and $h \in H_j$. This shows that $H \cap \pi_j H = H_j$ for $j \leq 4$. We may assume that π_j are unit elements, so for a basic element $e \in M^j$, we have $e\pi_j \in H_j$ and $e\pi_j A$ is pure in H_j. It follows that $e\widehat{A} \cap H_j = e\pi_j A$.

Finally, consider $\phi \in \operatorname{End}_R H$. By continuity, ϕ admits a unique extension to an endomorphism $\widehat{\phi}$ of $\widehat{M}$ which is an $\widehat{R}$-homomorphism. We now consider $\widetilde{\phi} = \widehat{\phi} \restriction \widetilde{M}$. If we write $M = \bigoplus_{i \in \lambda} e_i A$, then $\widetilde{M} = \bigoplus_{i \in \lambda} e_i \widehat{A}$, so $M \subset H \subset \widetilde{M}$ implies that $e_i \phi \in \widetilde{M}$. From $\widehat{A} = \widehat{R}$ it follows for the $\widehat{R}$-homomorphism $\widetilde{\phi}$ that $e\widehat{A}\widetilde{\phi} \subseteq e\widetilde{\phi}\widehat{A}$, and hence we obtain $\widetilde{M}\widetilde{\phi} \subseteq \widetilde{M}$.

From our first argument it follows that $H_j \phi \subseteq H_j$ for all $j \leq 4$. Thus $\widetilde{H}_j \widetilde{\phi} \subseteq \widetilde{H}_j$ by topological closure. This implies $\phi = a \in \widehat{A}$ in view of $\operatorname{End} \widetilde{\mathbf{M}} = \widetilde{A}$. Now pick $e\pi_1$ as above. Then $e\pi_1 \widetilde{\phi} = e\pi_1 a \in e\widehat{A} \cap H_1 = e\pi_1 A$, and therefore also $\widetilde{\phi} \in A$, completing the proof. ∎

We now illustrate cases where the hypotheses of the preceding theorem are satisfied.

Case E: Let $R = \mathbb{Z}$, furnished with the p-adic topology for a prime number p. In the ring J_p of the p-adic integers choose a pure subring $A \supseteq \mathbb{Z}_p$ such that there are at least 5 algebraically independent elements (in J_p) over A — this condition is certainly satisfied whenever A is countable. In view of the preceding theorem we can state

Corollary 5.8: *Let A be a pure subring of the p-adic integers as stated. For every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely A-rigid family of A-modules M_U $(U \subseteq \lambda)$ of cardinality λ.*

Case F: Let R be a commutative ring of cardinality $< 2^{\aleph_0}$, and S a countable multiplicatively closed subset of regular elements of R. Assume A is an R-algebra with $|A| < 2^{\aleph_0}$ that is S-reduced (i.e., $\bigcap_{s\in S} sA = 0$) and S-torsion-free (i.e., $sa = 0$ for some $s \in S, a \in A$ implies $a = 0$) such that A is an S-pure submodule between R and the S-completion $\widetilde{R}$ of R. It follows from Göbel–May [13, p. 216, Theorem] that the S-completion $\widetilde{R}$ of R contains elements π_γ $(\gamma < 2^{\aleph_0})$ that are algebraically independent over A. Consequently, we have

Corollary 5.9: *Let R, A be as stated. Then for every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely A-rigid family of A-modules M_U $(U \subseteq \lambda)$ of cardinality λ.*

Case G: Suppose R is a quasi-local ring with maximal ideal P such that $\bigcap_{n<\omega} P^n = 0$. If the completion $\widetilde{R}$ of R in the P-adic topology contains at least 5 algebraically independent units over R, then Theorem 5.7 applies, and we are led to the following conclusion

Corollary 5.10: *Let R be a quasi-local ring as stated. Then for every infinite cardinal $\lambda < \kappa(\omega)$, there exists an absolutely R-rigid family of R-modules M_U $(U \subseteq \lambda)$ of cardinality λ.*

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